CAN THE DICE BE FAIR BY DYNAMICS?
J. STRZALKO, J. GRABSKI, A. STEFANSKI and T. KAPITANIAK
Division of Dynamics, Technical University of Lodz,
Stefanowskiego 1/15, 90-924 Lodz, Poland
Received June 9, 2009; Revised July 21, 2009

We consider the dynamics of the three-dimensional model of the die which can bounce with dissipation on the table. It is shown that for the realistic values of the initial energy the probabilities of the die landing on the face which is the lowest one at the beginning is larger than the probabilities of landing on any other face.

Keywords: Dice tossing; predictability; basins of attraction.

1. Introduction

A throw of a fair die is commonly considered as a paradigm for chance. The die is usually a cube of a homogeneous material. The symmetry suggests that such a die has the same chance of landing on each of its six faces after a vigorous roll so it is considered to be fair. Generally, a die with a shape of convex polyhedron is fair by symmetry if and only if it is symmetric with respect to all its faces [Diaconis & Keller, 1989]. The polyhedra with this property are called the isohedra. Every isohedron has an even number of faces [Grunbaum, 1960]. The commonly known examples of isohedra are: tetrahedron, octahedron, dodecahedron and icosahedron which are also used as the shapes for dice. Diaconis and Keller [1989] showed that there are not symmetric polyhedra which are fair by continuity. As an example, they considered the dual of n-prism which is a di-pyramid with 2n identical triangular faces from which two tips have been cut with two planes parallel to the base and equidistant from it. If the cuts are close to the tips, the solid has a very small probability of landing on one of two tiny new faces. However, if the cuts are near the base, the probability of landing on them is high. Therefore by continuity, there will be cuts for which new and old faces have equal probability. In [Diaconis & Keller, 1989] it is suggested that the locations of these cuts depend upon the mechanical properties of the die and the table and can be found either experimentally or by the analysis based on the classical mechanics.

However, these definitions are not considering the dynamics of the die motion during the throw. This dynamics is described by perfectly deterministic laws of classical mechanics which map initial conditions (position, configuration, momentum and angular momentum) at the beginning of the motion into one of the final configurations defined by the number on the face on which the die lands. From the point of view of the dynamical systems the outcome from the die throw is deterministic, but as the initial condition–final configuration mapping is strongly nonlinear, one can expect deterministic unpredictability due to the sensitive dependence on the initial conditions and fractal boundaries between the basins of different final configurations.

The implementation of this idea has been carried out in a few papers which deal mainly with the coin tossing problem [Ford, 1983; Zeng-Yung, 1985; Keller, 1986; Vulovic & Prange, 1986; Kechen, 1990; Balzass et al., 1995; Murray & Teare, 1993; Diaconis et al., 2007]. In these works the consideration of the various simplified models of the coin motion (for a detailed discussion see [Strzalk et al., 2008]) lead to the conclusion that the uncertainties of the outcome are due to the inability of setting
the initial conditions in a sufficiently accurate way (for the precise initial conditions the coin tossing is predictable). In our previous work [Strzalko et al., 2008] we studied the full three-dimensional model of the coin which considers the nonhomogeneous material of the coin, the air resistance, the elasticity and the friction of the floor on which the coin bounces. We showed that the process of the coin bouncing on the floor has a significant influence on the final configuration (heads or tails) as the successive impacts introduce sensitive dependence on initial conditions leading to transient chaotic behavior. With the increase of the number of bounces, the structure of the basin boundaries becomes more complex so the distance of a typical initial condition from a basin boundary is so small that practically any uncertainty in initial conditions can lead to the uncertainty of the results of tossing. In theory, the result of the coin tossing is unpredictable only in the case of infinite bounces. However, for the realistic material of the coin and the floor, only a few bounces can be observed (usually less than ten) so transient chaos cannot be well developed. Finally, one should mention the works of Feldberg et al. [1990] and Nagler and Richter [2008] which considered the dynamics of the simple die model, namely a barbell with two marked masses at each end that is thrown with initial rotation. Their analysis of the plane motion, which includes bouncing on the flat, horizontal table, points the connection between the number of the bounces and the predictability of the outcome.

In this paper, we consider the dynamics of the three-dimensional model of the die. We show that the probability of the die landing on the face, which is the lowest one at the beginning is larger than on any other face. The probabilities of landing on any face approach the same value 1/6 as the number of die bounces on the table. This case cannot be realized in the experiment due to the limitations of the initial energy (particularly when the die is thrown from the hand or cup) and dissipation of the energy during the bounces. This allows us to draw the conclusion that dice are not fair by considering the dice with isohedral shape and sharp edges and corners. (Precision casino dice have their pips drilled, and they are filled flush with a paint of the same density as the acetate used for the dice, so they remain in balance. They also have sharp edges and corners. (See the Wikipedia article on dice at http://en.wikipedia.org/wiki/Dice.) The typical examples of such dice are: tetrahedron, cube, octahedron, dodecahedron, icosahedron. We neglected the influence of air resistance, as in [Zeng-Yuan & Bin, 1985; Strzalko et al., 2008] it has been shown that the influence of the air resistance on the motion of the tossed coin is very small.

To describe the motion of the die in three-dimensional space, we introduce the body embedded frame (C′ξηζ). It is possible to show that the axis ξ,η and ζ as well as any other axis passing through point C are principal axis so the considered die models are spherical tops (e.g. [Landau & Lifschitz, 1976; Goldstein, 1950]). Their moments of inertia for any axis passing through C are equal to J (i.e. Jξ = Jη = Jζ = J) and deviation moments Jξη, Jξζ and Jηζ are equal to zero. Using the numerical procedures for moments and products of inertia of polyhedron bodies proposed in [Mirtich, 1996] we get J = ma^2/20 for tetrahedron, J = ma^2/6 for cube, J = ma^2/10 for octahedron, J = (ma^2/300)(95 + 39√5) for dodecahedron and J = (ma^2/20)(3 + √5) for icosahedron, where a is the length of the dice edge.

Consider the die motion in three-dimensional space described by the fixed frame 0xyz as shown in Fig. 1(a). Neglecting the influence of the air resistance one obtains Newton–Euler equations of motion in the following form

\[ m\ddot{x} = 0, \quad m\ddot{y} = 0, \quad m\ddot{z} = -mg, \]  \( (1) \)

\[ J_\xi\ddot{\xi} = 0, \quad J_\eta\ddot{\eta} = 0, \quad J_\zeta\ddot{\zeta} = 0, \]  \( (2) \)

where (\( \omega_\xi, \omega_\eta, \omega_\zeta \)) are the components of the angular velocity vector \( \omega \).

The integrals of Eqs. (1) and (2) are given in the form:

\[ \dot{x} = v_0x = \text{const}, \quad \dot{y} = v_0y = \text{const}, \quad \dot{z} = -gt + v_0z, \]  \( (3) \)

\[ \omega_\xi = \omega_\eta = \text{const}, \quad \omega_\zeta = \omega_\xi = \omega_\eta = \text{const}, \quad \omega_\zeta = \omega_\xi = \omega_\eta = \text{const}. \]  \( (4) \)

Equations (3) and (4) show that during the motion, the angular velocity of the die \( \omega \) is constant and
equal to its initial value $\omega_0$. The orientation of the vector $\omega$ in relation to the frames $C'\xi'\eta'\zeta'$ and $Oxyz$ is not changing as the components of angular velocity vector are constant (4). The directions of the angular velocity vector $\omega$ and the angular momentum vector $K_C$ coincide and are constant during the motion as shown in Fig. 1(a) and depend only on initial conditions.

The initial conditions are given by the initial position of the center of die mass $C - q_{0} = [x_0, y_0, z_0]^T$, its initial velocity $v_0 = [v_{x0}, v_{y0}, v_{z0}]^T$, initial orientation of the axis $(C'\xi'\eta'\zeta' - q_{20}) = [\psi_0, \theta_0, \phi_0]^T$ (given by the Euler angles) and initial angular velocity $\omega_0 = [\omega_{x0}, \omega_{y0}, \omega_{z0}]^T$.

To determine on which face the die lands, we analyze the full rotation of the die, i.e. $\varphi_i < 0$, $2\pi > \varphi > 0$ around the given rotation axis. In the dice games, usually the top face counts. We consider the lowest face (the face on which die lands) as one of the considered die shapes is tetrahedron which has no top face. During the die motion the projections of the vectors $n_i^b$ ($i = 1, \ldots, n$) (as shown in Fig. 1(b)) on the vertical axis $-k^v$ (with the orientation towards the table) have been calculated. This allows the calculation of the cosines of the angles $\alpha_i$: $\cos\alpha_i = -k^v \cdot n_i^b$. The face $j$ for which $\cos\alpha_j$ is the largest, i.e. $\cos\alpha_j = \max_i \{\cos\alpha_i, (i = 1, \ldots, n)\}$ is the lowest. The die lands on the face which is the lowest at the moment when the die stops on the table.

To describe a collision of the die with a table we assume that: (i) the table is modeled as flat, horizontal, elastic body (fixed to move), (ii) a friction force between the table and the die is omitted, (iii) only one point of the die is in contact with the table during each collision. Let us consider that the die collides with a table when the vertex $A$ touches the floor as shown in Fig. 1(c). According with Newton’s hypothesis one gets $\chi = v_{Al}/v_{A0}$, where $\chi$ is the coefficient of restitution, $v_{A0}$ and $v_{Al}$ are the projections of the velocity of the point $A$ on the direction $(z)$ normal to the impact surface, before and after the impact, respectively. The position of the point $A$ in the body embedded frame is described by $\xi_4, \eta_4, \zeta_4$. To describe the impacts we consider an additional frame with an origin at point $A$ and the axes: $x'\ y'\ z'$—parallel to the fixed axes $x\ y\ z$ [Fig. 1(c)]. In matrix form the velocity vector of the point $A$ is described as $v_4 = v_c + \Omega R \xi_4$, where: $v_c = [x\ y\ z]^T$, $\xi_4 = [\xi_4, \eta_4, \zeta_4]^T$, $\Omega = R^T \dot{R}$, and the transformation matrix (in terms of Euler angles)

$$R = \begin{bmatrix}
\cos\varphi \cos\psi - \cos\theta \sin\varphi \sin\psi & -\cos\psi \sin\varphi - \cos\theta \cos\varphi \sin\psi & \sin\varphi \sin\psi \\
\cos\psi \cos\varphi + \cos\theta \sin\varphi & -\cos\theta \sin\psi - \cos\varphi \sin\psi & \sin\varphi \cos\psi \\
\sin\varphi \sin\psi & \cos\varphi \sin\psi & \cos\varphi \cos\psi
\end{bmatrix}.$$ \hspace{1cm} (5)

In the analysis of the phenomena that accompany the impact besides Newtons hypothesis, the laws of linear momentum and angular momentum theorems of rigid body, as well as constraint equations have been employed. Modeling the nonholonomic contact between the die and the table, we consider the case of the smooth-frictionless die [Nejmark & Fufajev, 1972], i.e. the vector of the impulse base reaction $S$ has
the following form $S = [0, 0, S_x]$. The above assumptions result in the following relations:

$$
\begin{align*}
\dot{z} &= \frac{\partial}{\partial z}(-\zeta_4 \cos(\varphi + \psi) \sin \vartheta + \cos \vartheta \cos(\varphi + \psi)(\zeta_4 \cos \varphi + \zeta_4 \sin \varphi) + (\zeta_4 \cos \varphi - \zeta_4 \sin \varphi) \sin(\varphi + \psi)) \\
+ \dot{y} &= \frac{\partial}{\partial y} \left( \zeta_4 \cos \vartheta \cos \psi \sin^2 \varphi + \zeta_4 \cos^2 \psi \cos \vartheta \sin(2\varphi) - \zeta_4 \sin^2 \varphi \sin \psi \right) \\
+ \xi_4 \cos 2 \vartheta \cos(2\varphi + \psi) \sin \varphi + \cos^2 \varphi(-\zeta_4 \cos \psi + \zeta_4 \cos \varphi \sin \psi) - \zeta_4 \sin \varphi \sin(\varphi + \psi) \right) \\
= \chi \left( \dot{\varphi} + \dot{\varphi} \right) + \left( \zeta_4 \cos \vartheta \cos \psi \sin^2 \varphi + \zeta_4 \cos^2 \psi \cos \vartheta \sin(2\varphi) \\
- \zeta_4 \sin^2 \varphi \sin \psi + \zeta_4 \cos^2 \psi \sin(2\varphi) \right) \\
\end{align*}
$$

Equations (6)–(8) allow the determination of the die velocities components after the collision ($\dot{x}'$, $\dot{y}'$, $\dot{z}'$, $\omega_x'$, $\omega_y'$, $\omega_z'$) and the floor reaction impulse $S_x$.

The die motion after the collisions is given by Eqs. (1)–(3) with new initial velocities: $\dot{x}'$, $\dot{y}'$, $\dot{z}'$, $\omega_x'$, $\omega_y'$, $\omega_z'$ calculated from Eqs. (6)–(8) and the same initial positions $x, y, z, \varphi, \vartheta, \psi$ as before the impacts.

### 3. Results and Discussion

The examples of the trajectories of dice vertices calculated from Eqs. (1), (2), (6)–(8) are shown in Figs. 2(a)–2(d). In the numerical calculations we assume the following parameters: $m = 0.016$ [kg], $\chi = 0.5$, and initial conditions: $x_0 = y_0 = 0$, $v_{x0} = 0.6$ [m/s], $v_{y0} = 0$, $v_{z0} = -0.7$ [m/s], $\varphi_0 = 0.0001$ [rad], $\vartheta_0 = 0.0001$ [rad], $\psi_0 = 0$, $\omega_{x0} = 60$ [rad/s], $\omega_{y0} = 0$. Vertical lines indicate the position of the successive collisions. For the tetrahedron die $\alpha = 0.040793$ [m]) during the successive bounces the following corners collide with the table: $A, B, C, D, A, B, A, B, D$ as shown in Fig. 2(a). The cube die ($\alpha = 0.02$ [m]) hits the table with $C$, $D, D, C, B, D, B, B, B$ corners (Fig. 2(b)). The sequences of the colliding corners for the octahedron ($\alpha = 0.025698$ [m]) [Fig. 2(c)] and icosahedron ($\alpha = 0.015426$ [m]) dice are respectively $C, D, C, B, A, B, B, B$ [Fig. 2(c)] and $A, C, B, E, F, B, L, D, I$ [Fig. 2(d)]. Notice that not all collisions result in the change of the die face which is the lowest one before and after the collision. For the tetrahedron die we observe such a face change only once after the fourth collision. Four changes take place for the cube die (after third, fourth, fifth and seventh collisions). During the simulation of the throw of the octahedron and icosahedral die one observes respectively 3 (after first, second and third collisions) and 5 (after first, second, third, fifth and eighth collisions) face changes.

Figures 3(a)–3(d) describe the dissipation of the energy along the trajectories shown in Figs. 2(a)–2(d). Total mechanical energy $E_{\text{tot}} = 1/2m(v_x^2 + v_y^2 + v_z^2) + 1/2J(\omega_x^2 + \omega_y^2 + \omega_z^2) + mgz$ (green line), rotational kinetic energy $E_{\text{rot}} = 1/2J(\omega_x^2 + \omega_y^2 + \omega_z^2)$ (red line) and the sum of kinetic energy of translation and potential $E_{\text{tot}} = 1/2m(v_x^2 + v_y^2 + v_z^2) + mgz$ (black line) are shown. Blue lines indicate the level of the minimum energy necessary to change the throw result $E_{\text{min}} = 1/2m(v_x^2 + v_y^2 + v_z^2) + 1/2J(\omega_x^2 + \omega_y^2 + \omega_z^2)$, where $h_{\text{min}}$ is the distance between mass center and the die edge. Notice that although the total energy decreases with the successive collisions, after particular collisions rotational energy $E_{\text{rot}}$ can increase what results in the unexpected increase of rotation speed. Such a phenomena are easily visible in real (experimental) die throw.

For $n$ face die there are $n$ possible final configurations (the die can land on one of its faces $F_i$ ($i = 1, 2, \ldots, n$). All initial conditions are mapped.
into one of the final configurations. The initial conditions which are mapped onto the $i$th face configuration create $i$th face basin of attraction $\beta(F_i)$. The boundaries which separate the basins of different faces consist of initial conditions mapped onto the die standing on its edge configuration which is unstable. For the dice with sharp edges one can assume that the set of initial conditions which are mapped into the boundaries has a zero Lebesque measure.

The possibility that the boundaries between the basins of different faces are fractal [Grebogi et al., 1983], particularly riddled [Alexander et al., 1992; Sommerer & Ott, 1993; Ott et al., 1994; Kapitaniak et al., 1998, 1994; Kapitaniak, 1996] or intermingled [Alexander et al., 1992; Kapitaniak, 1996] ones is worth investigating. Near a given basin boundary, if the initial conditions are given with the uncertainty $\epsilon$, then a fraction $f(\epsilon)$ of initial conditions gives the unpredictable outcome. In the limit $\epsilon \to 0$, $f(\epsilon) \propto \epsilon^\alpha$ where $\alpha < 1$ for fractal and $\alpha = 1$ for smooth boundary. From the point of view of the predictability of the die throw the possibility of the occurrence of the intermingled basins is the most interesting. Let us briefly explain the term of the intermingled basins of attraction. A basin $\beta(F_i)$ is called the riddled one when: (i) it has a positive Lebesque measure, (ii) for any point in $\beta(F_i)$, a ball in the phase space of arbitrarily small radius has a nonzero fraction of its volume in some other (say $\beta(F_j)$) basin. The basin $\beta(F_j)$ may or may not be riddled by the basin $\beta(F_i)$. If the basin $\beta(F_j)$ is also riddled by the basin $\beta(F_i)$ such basins are called the intermingled ones. In the case of the thrown die the intermingled basins of attraction of all $n$ faces will mean, that in any neighborhood of the initial condition leading to one of $F_i$, there are initial conditions which are mapped to other faces, i.e. there does not exist an open set of initial conditions which is mapped to one of the final configurations or infinitely small inaccuracy in the initial conditions makes the result of the die throw unpredictable.
Analysis of the structure of the basin boundaries allow us to identify the condition under which the die throw is predictable and fair by dynamics.

**Definition 1.** The die throw is predictable if for almost all initial conditions $x_0$ there exists an open set $U (x_0 \in U)$ which is mapped into the given final configuration.

Assume that the initial condition $x_0$ is set with the inaccuracy $\epsilon$. Consider a ball $B$ centered at $x_0$ with a radius $\epsilon$. Definition 1.1 implies that if $B \subset U$ then randomizer is predictable.

**Definition 2.** The die throw is fair by dynamics if in the neighborhood of any initial condition leading to one of the $n$ final configurations $F_1, \ldots, F_n$, where $i = 1, \ldots, n$, there are sets of points $\beta(F_1), \ldots, \beta(F_n)$, which lead to all other possible configurations and a measures of sets $\beta(F_i)$ are equal.

Definition 2 implies that for the infinitely small inaccuracy of the initial conditions all final configurations are equally probable.

Figures 4(a)–4(f) and Figs. 5(a)–5(c) show the basins of attraction of different faces of the respectively tetrahedron and cube dice. They are based on the results obtained from numerically integrated equations of motion (1, 2, 6–8). We fixed all initial conditions except two namely: the position of the die mass center $z_0$ and the angular velocity $\omega_0$. We consider $m = 0.016$ [kg], $a = 0.040793$ [m] (tetrahedron), $m = 0.016$ [m], $a = 0.02$ [m] (cube) and the following initial conditions: $v_{x0} = v_{y0} = 0$, $v_{z0} = -0.7$ [m/s] $v_{x0} = 0.001$ [rad], $v_{y0} = 0.0001$ [rad], $\omega_{x0} = 0$, $\omega_{y0} = 0$, $\omega_{z0} = 60$ [rad/s]. The case of the tetrahedron die terminating on the soft table surface ($\chi = 0.6$) is shown in Figs. 4(d)–4(f). The basins of attraction of the cube die are shown in Figs. 5(a)–5(c). The die bounce on the table surface is characterized by the restitution coefficient $\chi = 0.6$. The
basins of attraction of different die faces: 1, 2, 3, 4, 5 and 6 face are shown respectively in blue, green, yellow, red, brown and navy blue.

We observe that similar structures of the basins boundaries are seen when different initial conditions are allowed to vary, so Figs. 4(a)–4(f) and Figs. 5(a)–5(c) represent the two-dimensional sections of the phase space as good indications of what occurs in the entire phase space.

The structure of the basins boundaries for the case without bouncing on the table [Figs. 4(a)–4(c)] are similar to the boundaries previously obtained for the coin tossing models [Keller, 1986; Strzalko et al., 2008]. One notices that the structure of the basins boundaries is more complicated (looks like fractal or intermingled) when the die is allowed to bounce on the table as it can be seen in Figs. 4(d) and 5(a). To check the possibility that these basins are fractal (intermingled) the appropriate enlargements are presented in Figs. 4(e) and 4(f), Figs. 5(b) and 5(c). It can be seen that apart from the graininess due to the finite number of
Fig. 5. Basins of attraction of different cube die faces; basins of 1, 2, 3, 4, 5 and 6 face are shown respectively in blue, green, yellow, red, brown and navy blue. Eqs. (1), (2), (6)–(8) have been integrated for the following initial conditions: \( x_0 = y_0 = 0 \), \( v_{0x} = v_{0y} = v_{0z} = 0 \), \( \psi_0 = 0.3 \) [rad], \( \theta_0 = 1.2 \) [rad], \( \phi_0 = 0.6 \) [rad], \( v_{0x} = 0 \), \( v_{0y} = 0 \), \( v_{0z} = 0 \), \( \psi_0 = 0.3 \) [rad], \( \theta_0 = 1.2 \) [rad], \( \phi_0 = 0.6 \) [rad], \( \omega_{0x} = 0 \), and \( \omega_{0y} = 0 \), landing on the surface characterized by the restitution coefficient \( \chi = 0 \), (b) and (c) present enlargements of (a).

Fig. 6. Basins of attraction in \( z_0 - \omega_{0y} \) plane for the tetrahedron die; basins of 1, 2, 3 and 4 face are shown respectively in blue, green, yellow and red. Eqs. (1), (2), (6)–(8) have been integrated for the following initial conditions: \( x_0 = y_0 = 0 \), \( v_{0x} = v_{0y} = v_{0z} = 0 \), \( \psi_0 = 0.3 \) [rad], \( \theta_0 = 1.2 \) [rad], \( \phi_0 = 0.6 \) [rad], \( v_{0x} = 0 \), \( v_{0y} = 0 \), \( v_{0z} = 0 \), \( \psi_0 = 0.3 \) [rad], \( \theta_0 = 1.2 \) [rad], \( \phi_0 = 0.6 \) [rad], \( \omega_{0x} = 0 \), and \( \omega_{0y} = 0 \). (a) \( \chi = 0 \), (b) \( \chi = 0.5 \), (c) \( \chi = 0.5 \), (d) \( \chi = 1 \).
points, the boundaries are smooth. Under further magnification no new structure can be resolved, i.e. no evidence of intermingled or even fractal basin boundaries is visible. The same conclusion have been reached in [Vulovic & Prange, 1986; Mizuguchi & Suwashita, 2006; Kechen 1990] for simple one-dimensional models of the coin tossing.

This allows us to state that the result of the die throw is predictable according to Definition 1. In other words, if one can settle the initial condition with appropriate accuracy, the outcome of the coin tossing procedure is predictable and repeatable. When the die is thrown from the hand or the cup the realistic values of the restitution coefficient $\chi$ equal respectively to 0.05, 0.2, 0.5 and 1. The basin boundaries shown in Figs. 6(a)–6(c) are smooth. With the increase of the restitution coefficient $\chi$ (lower energy dissipation), it is possible to observe that the complexity of the basin boundaries increases. A similar mechanism of fractalization has been observed for the tossed coin [Strzalko et al., 2008]. The same properties of the basin boundaries have been observed for several cases of different initial conditions as well as for the dice with different shape (we consider tetrahedron, octahedron, dodecahedron, icosahedron shapes). Figure 6(d) presents the result obtained for the unrealistic case of $\chi = 1$, i.e. elastic collisions with no energy dissipation. One observes the face of the die which is the lowest after the 1000th collision. For this case, the series of die vertices which collide with the table in the successive bounces B, A, A, C, D, D, A, B, C, C, B, A, D, B, B, B, C, A, A, B, D, D, C, C, . . . is chaotic. The largest Lyapunov exponent estimated from this time series is equal to 0.076. It seems that only in this case the basins are intermingled (up to the numerical accuracy) and the results of the die throw are unpredictable.

We perform the following numerical experiment. For a given value of $\omega_0$ and fixed initial conditions $x_0 = y_0 = v_x = v_y = v_{\xi_0} = \omega_{\xi_0} = \omega_0 = 0$ we randomly choose $2 \cdot 10^6$ values of the rest of initial conditions from the following set: $z_0 \in [15\pi, 20\pi]$, $v_0 \in [0, 2\pi]$, $\theta_0 \in [0, 2\pi]$, $\phi_0 \in [0, 2\pi]$ and integrate Eqs. (1), (2), (6)–(8). Let $(p^*)$ be the average probability that the die lands on the face which is the lowest one at the start. In Table 1 we have presented the difference between $(p^*)$ and the theoretically expected value of this probability $1/n$ for different values of $\omega_0$. In this experiment, $\omega_0$ gives initial rotation energy of the system $E_{\text{rot}} = (1/2)J\omega_0^2$. Dice with tetrahedron, cube and octahedron shapes have been considered. One can observe that the probability $(p^*)$ approaches the theoretical values $1/n$ for large values of $\omega_0$. In such cases, the die typically bounces on the table between 15 and 25 times before it no longer can change its orientation. When the die is thrown from the hand or the cup the realistic values of $\omega_0$ are 20–40 [rad/s] (for vigorous throw) and the typical number of bounces is about 4–5. From Table 1, one

<table>
<thead>
<tr>
<th>$\omega_0$ [rad/s]</th>
<th>Tetrahedron</th>
<th>Cube</th>
<th>Octahedron</th>
<th>Icosahedron</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.393</td>
<td>0.217</td>
<td>0.212</td>
<td>0.117</td>
</tr>
<tr>
<td>10</td>
<td>0.341</td>
<td>0.142</td>
<td>0.133</td>
<td>0.098</td>
</tr>
<tr>
<td>20</td>
<td>0.282</td>
<td>0.101</td>
<td>0.081</td>
<td>0.043</td>
</tr>
<tr>
<td>30</td>
<td>0.201</td>
<td>0.065</td>
<td>0.066</td>
<td>0.038</td>
</tr>
<tr>
<td>40</td>
<td>0.092</td>
<td>0.063</td>
<td>0.029</td>
<td>0.018</td>
</tr>
<tr>
<td>50</td>
<td>0.073</td>
<td>0.022</td>
<td>0.024</td>
<td>0.012</td>
</tr>
<tr>
<td>100</td>
<td>0.052</td>
<td>0.013</td>
<td>0.015</td>
<td>0.004</td>
</tr>
<tr>
<td>200</td>
<td>0.009</td>
<td>0.008</td>
<td>0.007</td>
<td>0.002</td>
</tr>
<tr>
<td>300</td>
<td>0.005</td>
<td>0.005</td>
<td>0.003</td>
<td>0.001</td>
</tr>
<tr>
<td>1000</td>
<td>0.003</td>
<td>0.002</td>
<td>0.001</td>
<td>0.000</td>
</tr>
</tbody>
</table>
can see that in this case the die face which is the lowest at the beginning is significantly more probable than other faces, so in the realistic mechanical experiment, the dice are not fair. It is not enough for a die which is fair by symmetry to be fair by dynamics and by mechanical experiments or simulations one cannot construct the die which is fair by continuity.

4. Conclusions
To summarize, in this paper we consider the dynamics of the three-dimensional model of the die of different isohedral shapes. We show that the probability that the die lands on the face which is the lowest is larger than on any other face, i.e. the die is not fair by the dynamics. If an experienced player can reproduce the initial conditions with a small finite uncertainty, there is a good chance that the desired final state will be obtained. The probabilities of landing on any face approach the same value $1/n$ only for large values of the initial rotational energy and a great number of die bounces on the table. This can be done in computer simulations but not in the real experiment when a die is thrown from the hand or the cup as due to the limitation of the initial energy, the die can bounce only a few times. Theoretically, probabilities of landing on any face are equal only in the unrealistic Hamiltonian case of infinite number of die bounces on the table when the dynamics is chaotic.

Acknowledgment
This study has been supported by the Polish Department for Scientific Research (DBN) under project No. NN501 0710 33.

References