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Controlling chaos by chaos in geophysical systems

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Abstract. Using the Lorenz equations as an example we show that one chaotic system can be controlled by synchronizing its behavior with the chaotic behavior of another system. We particularly discuss the implications of this phenomenon in geophysical systems.

The difficulty of carrying out long-term predictions of atmospheric dynamics and the evolution of climate is a problem of obvious concern. Nowadays there is increasing awareness that deterministic chaos might provide a possible paradigm for the complexity of atmospheric and climatic dynamics. Periodicity is not the first apparent characteristic of the behavior of many geophysical fluid dynamic systems, but atmospheric and oceanic flows often exhibit substantial coherent features, localized in either or both of space and time, which occur sporadically and unpredictably but with a certain statistical regularity which can be important in extended-range atmospheric prediction. Such features are exemplified by blocking patterns in the mid-latitude atmosphere, or by persistent anomalies of the ocean-atmosphere system (of which El Nino is the spectacular example [McCreary and Andeson, 1991, Brindley et al., 1992]); their presence coincides with temporary and localized improvement in potential predictability.

In this letter we propose a mechanism for reduction in chaos which could affect atmospheric potential predictability based on the continuous chaos control scheme [Pyragas, 1992, 1993, Qu et al., 1993].

We consider two chaotic systems, which we call A and B respectively,

$$\dot{x} = f(x) \qquad \dot{y} = g(y) \tag{1}$$

where $x, y \epsilon \mathbf{R}^n$, and we use the controlling strategy which is schematically illustrated in Figure 1; the two systems are coupled through the operators λ, μ, which we take to have a very simple linear form. We assume that some or all state variables of both systems A and B can be measured, so that we can measure signal $x(t)$ from the system A and signal $y(t)$ from B, and that the systems are coupled in such a way that the differences $D_{1,2}(t)$ between the signals $x(t)$ and $y(t)$ are used as control signals

$$\begin{aligned} F_1(t) &= \lambda[x(t) - y(t)] = \lambda D_1(t) \\ F_2(t) &= \mu[y(t) - x(t)] = \mu D_2(t) \end{aligned} \tag{2}$$

introduced respectively into each of the chaotic systems A and B as a negative feedback. We take λ, $\mu > 0$ to be experimentally adjustable weightings of the perturbation.

Using the coupling schematically shown in Figure 1 we have shown that one chaotic system coupled with the other one can significantly change the behavior of one of them (unidirectional coupling, i.e., λ or $\mu = 0$) or of both systems (mutual coupling, i.e., λ, $\mu \neq 0$). This property allows us to describe the above procedure as the controlling chaos by chaos method.

Propositions 1 and 2 presented in the Appendix give rigorous conditions under which chaotic attractors of systems A and B are equivalent (Proposition 1), or the evolution of one of them is forced to take place on the attractor of the other one (Proposition 2). Detailed investigation of the question of equivalence of chaotic attractors is given elsewhere [Kocarev and Kapitaniak, 1994]. Here we describe some applications of controlling chaos by chaos in geophysical systems, pointing out that, even when the conditions of Propositions 1 and 2 are not fulfilled the introduction of coupling can still have practical importance.

In our numerical examples we first consider two Lorenz models [Lorenz, 1963, 1965] mutually coupled in the following way

$$\begin{aligned} \dot{X}_1 &= -\sigma X_1 + \sigma Y_1 + \lambda(X_2 - X_1) \\ \dot{Y}_1 &= -X_1 - Z_1 + r_1 \ X_1 - Y_1 + \lambda(Y_2 - Y_1) \\ \dot{Z}_1 &= X_1 \ Y_1 - bZ_1 + \lambda(Z_2 - Z_1) \\ \dot{X}_2 &= -\sigma X_2 + \sigma Y_2 + \mu(X_1 - X_2) \\ \dot{Y}_2 &= -X_2 - Z_2 + r_2 X_2 + \mu(Y_1 - Y_2) \\ \dot{Z}_2 &= X_2 - Y_2 - bZ_2 + \mu(Z_1 - Z_2) \end{aligned} \tag{3}$$

where σ, $r_{1,2}$ and b are constants. The Lorenz model has often been proposed as a paradigm for the "chaotic" extra-

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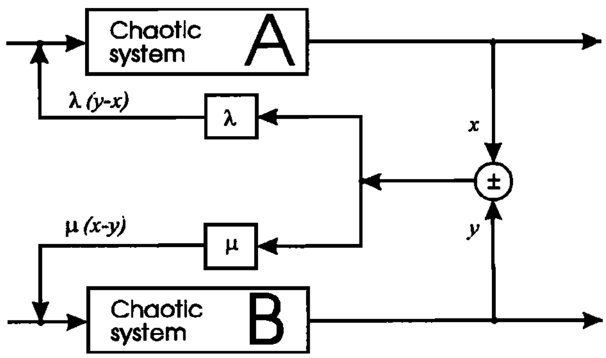

Figure 1. Scheme of controlling procedure.

tropical atmospheric circulation [Palmer, 1993]. The variables X, Y and Z then represent in some broad sense Rossby wave components of the extratropical general circulation. Coupling between two Lorenz models introduced in eq. (3) might then be interpreted as mutual interdependence of extratropical circulations in two regions characterized by different r parameter value, say an intensive storm track and a relatively stable anticyclonic region. The concept of teleconnections of this kind, achieved through the mechanism of quasi-linear Rossby trains, has both theoretical and observational support [Madden and Julian, 1971, Weickmann, 1991].

Numerical computations have been carried out using software INSITE [Parker and Chua, 1989]. In Figure 2 (a-b) we show the chaotic attractors of single Lorenz models ($\lambda,\mu=0$) for $\sigma=10.0$, $r_1=197.4$, $b=8/3$ (Figure 2(a)) and $r_2=211.0$ (Figure 2(b)). These attractors are characterized by the following spectra of Lyapunov exponents $\lambda_1=1.87$, $\lambda_2=0$, $\lambda_3=-15.54$ (Figure 2(a)) and $\lambda_1=0.78$, $\lambda_2=0, \lambda_3=-14.44$. In Figure 2(c) we show the behavior of both above mentioned Lorenz systems coupled with $\lambda=100$ and $\mu=1$. Although this attractor is still chaotic ($\lambda_1=0.79$, $\lambda_2=0$, $\lambda_3=-14.34$), trajectory behavior on it is more predictable as its Lyapunov dimension, $d_L=2.053$, is smaller than the dimension the of original attractor ($d_L=2.121$). This dimension increase is produced by a significant decrease of positive Lyapunov exponent ($\lambda_1=0.79$ in comparison with $\lambda_1=1.87$ of the original attractor).

In a second example we consider the coupling of a Lorenz system with a linear oscillator

$$\begin{aligned} \dot{W} &= -\Omega(t)V - k(W - W^*) \\ \dot{V} &= \Omega(t)(W - W^*) - kV \end{aligned} \tag{4}$$

which in geophysical context represents the tropical atmosphere [Palmer, 1993, Madden and Julian, 1971, Weickmann, 1991]. Here, Ω is taken to be the frequency of some dominant internal mode of large-scale variability of the tropics, e.g., the Madden-Julian oscillations. In our computations we consider Ω to be a time dependent random variable with uniform distribution in the interval [1.3, 1.7] in nondimensional time. V and W represent two phase-quadrature components of the tropical oscillations. For example W can be considered as representing the Walker circulation [Palmer, 1993].

Although eq. (4) is stochastic, any particular phase space trajectory has the properties of a chaotic trajectory, so we can apply system (4) to control the chaotic behavior of a Lorenz model, in the same way as before.

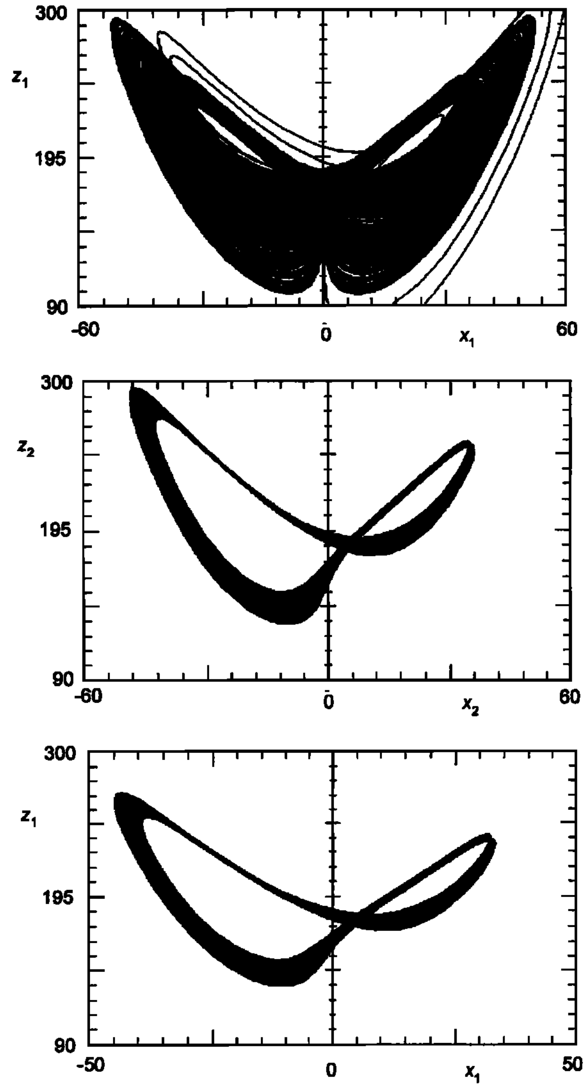

Figure 2. Evolution of Lorenz system; (a) unmodifird first system: $\sigma=10.0$, $r=197.4$, $b=8/3$; (b) unmodified second system: $\sigma=10.0$, $r=212.0$, $b=8/3$; (c) attractor of the first system controlled by the second system: $\lambda=100.0$, $\mu=1.0$.

Application of our chaos control method requires consideration of the following coupled equations:

$$\begin{aligned} \dot{X} &= -\sigma X + \sigma Y \\ \dot{Y} &= -XZ + rX - Y + \lambda(W - Y) \\ \dot{Z} &= XY - bZ \\ \dot{W} &= -\Omega V - k(W - W^*) \\ \dot{V} &= \Omega(t) - (W - W^*) - kV \end{aligned} \tag{5}$$

Examples of numerical calculations for $\sigma=10.0$, $r=197.5$, $b=8/3$ and $W^*=0$ are shown in Figure 3. Previously in Figure 2(a) we showed the original attractor of a Lorenz system given by eq. (3) with λ, $\mu=0$ (or eq.(5) with $\lambda=0$),

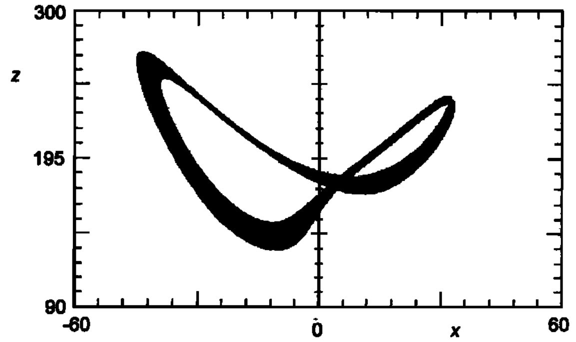

Figure 3. Lorenz attractor controlled by random trajectory of eq. (4): $\Omega\epsilon(1.3, 1.7)$, $W^*=0$ and $\lambda=10.0$.

while in Figure 3 the same attractor, modified by coupling the Lorenz system with eq. (4) with $\lambda=10.0$, is presented. Although the attractor of Figure 3 is chaotic, its dimension is smaller than the dimension of the original chaotic attractor of Figure 2(a). Thus the dynamics of the modified system (5) is far more predictable than the dynamics of a single Lorenz model. Its Lyapunov dimension estimated from time series by Wolf's et al. [1985] methods is $d_L=2.008$. This dimension decrease is again produced by significant reduction of positive Lyapunov exponent ($\lambda_1=0.69$ in comparison with $\lambda_1=1.87$ of the original attractor).

We would like to note that, despite the fact that in neither of our numerical examples the conditions of Propositions 1 and 2 have been precisely fulfilled, nevertheless application of our controlling procedure has allowed us to convert one type of chaotic behavior to chaotic behavior which is more predictable as we choose parameters λ and μ close to those that fulfill the above mentioned conditions. For λ and μ which are far away from the conditions of Propositions 1 and 2 our controlling procedure does not work.

In the cases considered in this letter the application of a continuous control scheme did not result in obtaining periodic behavior because the external periodic perturbation was not taken in the form of an unstable periodic orbit of original chaotic system. If that is done, a similar controlling scheme allows us to convert the original chaotic behavior into an appropriate periodic one [Pyragas, 1992, 1993, Qu et al., 1993]. It should be mentioned here that, if f=g and $\mu=0$ in eqs. (1), our controlling procedure simplifies the method of synchronization of chaos using continuous control [Kapitaniak, 1994].

In summary we propose that this procedure of controlling chaos by chaos can be treated as a possible mechanism for the so-called extended range atmospheric predictability observed in geophysical systems. The results of this simple coupling show the great potential influence of the behavior of chaotic system on that of another. It is also clear that, in a stationary state, when the variables x and y are close together, the control signals (2) are small (under conditions of Proposition 2 they converge to each other). This probably means that such a coupling might not allow for easy experimental verification in real geophysical systems. It is, however, a mechanism which may account for the existence of many unexpected short and relatively predicted features in the situations when strongly chaotic behavior might be expected. We hope that the results of this letter could be of direct benefit concerning the problem of "predicting predictability," which is currently a research topic of great interest.

Finally, we remark that a similar chaos by chaos controlling mechanism, having a different type of coupling, was recently proposed to be responsible for stabilization of the Earth's obliquity by the Moon [Laskar et al., 1993].

Appendix

Our controlling strategy results in the following dynamical system

$$\begin{aligned} \dot{x} &= f(x) + \lambda(y-x) \\ \dot{y} &= g(y) + \mu(x-y). \end{aligned} \tag{A1}$$

where λ, μ are real nonnegative parameters. Note that many systems can be put in the form of (A1) including the formulation [Smale, 1967] of the Turing reaction-diffusion theory [Turing, 1953] of morphogenesis or the evolution of two resistively coupled electrical circuits [Kapitaniak et al., 1993].

We assume that the dynamical systems

$$\begin{aligned} \dot{x} &= \mathrm{f}(x) \\ \dot{y} &= \mathrm{g}(y), \end{aligned} \tag{A2}$$

where $x,y \epsilon \mathbf{R}^3$ and (A2) have chaotic attractors A_f, A_g and A respectively. Denote the projection of A on the subspace $x=(x_1,x_2,x_3)^T$ by A_x, and the subspace $y=(y_1,y_2,y_3)^T$ by A_y.

Recalling the definition of topological equivalence of two chaotic attractors: namely an attractor A_f is equivalent to attractor A_g if there exists a homeomorphism h: $\mathbf{R}^n \to \mathbf{R}^n$ such that $h(A_f)=A_g$ we have the following propositions

Proposition 1.

(i) If f=g and

$$| x(t=0) - y(t=0) |$$

is sufficiently small, then there exists a value of $k=\lambda+\mu$, say k_*, such that for $k>k_*$, A_x is equivalent to A_y.

(ii) If f$\neq$g and $\lambda=\infty$, then A_x is equivalent to A_g.

(iii) If f$\neq$g and $\mu=\infty$, then A_y is equivalent to A_f.

Proof:

(i) First note that the inequalities

$$| a_{jj} | > \sum_{i=1,\, i\neq j}^{n} | a_{ij} |$$

where $j=1,\ldots,n$, are sufficient for the stability of a matrix $[a_{ij}]$ with negative diagonal elements.

Denote $u=x$-y, so that from (A1) we have

$$\begin{aligned} \dot{u} &= \left[-(\lambda+\mu)E + Df|_{u=0} \right] u + O(x,y) \equiv \\ &\equiv Au + O(x,y) \end{aligned}$$

where Df is the Jacobian matrix of f, E is the unit matrix and $O(x,y)$ represents the higher order terms. It is obvious that one can find k such that matrix $A=[a_{ij}]$ is stable, that is $u=0$ is asymptotically stable, and $x(t)$ approaches $y(t)$ as $t \to \infty$. Hence A_x is equivalent to A_y (the homeomorphism $h: \mathbf{R}^n \to \mathbf{R}^n$ is identity).

(ii) Equation (A1) can be rewritten as

$$\epsilon \dot{x} = \epsilon f(x) + (y-x)$$
$$\dot{y} = g(y) + \mu(x-y),$$

where $\epsilon = 1/\lambda$. If $\epsilon=0$, the last equation is equivalent to

$$x = y$$
$$\dot{y} = g(y).$$

Thus, A_x is equivalent to A_g (again, the homeomorphism $h: \mathbf{R}^n \to \mathbf{R}^n$ is identity).

(iii) The proof is similar as in the case (ii).

The second and the third part of the Proposition 1 can be improved in the following way:

Proposition 2.

For sufficiently small

$$| x(t = 0) + y(t = 0) |$$

and ϵ there exists t_0 such that $x(t)$ converges uniformly to $y(t)$ as $\epsilon \to 0^+$ on all subsets of $t_0 < t < \infty$.

Proof: The proof is similar to the proof of Theorem 2 in [Kocarev and Kapitaniak, 1994].

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