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Spatio–Temporal Chaos in Closed and Open Systems

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Abstract—Spatio–temporal chaos (temporal chaos coupled with spatial variability) widely occurs in turbulent phenomena and is associated with spatial pattern formation. In this review we address classical examples of spatio–temporal complexity, develop ideas in the context of coupled map lattices and speculate on possible quantifiers for spatio–temporal chaos.

1. INTRODUCTION: WHAT IS SPATIO–TEMPORAL CHAOS?

By spatio-temporal chaos (STC) we mean temporal high-dimensional chaos associated with spatial pattern dynamics. It occurs widely in turbulent phenomena, including Rayleigh–Bernard convection, electrically driven convection in liquid crystals, boiling, combustion, MHD turbulence in plasma, solid-state physics (Josephson junctions, spin wave turbulence), optics, chemical reactions with spatial structure, and so on. It is also important in biological information processing involving nonlinear characteristics, for example, neural dynamics. Although there is no clear definition for STC, we assume that the number of degrees-of-freedom is large and that the dimension (in phase-space) increases with the system's size.

In the purely temporal case, studies on low-dimensional chaos have expanded rapidly, and some understanding has developed. Dynamics of ordered spatial structure has been studied in pattern formation. However, though phenomena complex in both space and time are common in nature, little basic understanding has yet been developed. We are at an exploratory stage, seeking new phenomenology in a jungle of spatio–temporal chaos. Understanding the phenomenology may still require much time, but we present evidence in Section 5 of a structured hierarchy of qualitative behaviour, which gives support to the idea of at least some universality classes of STC.

As a preliminary, we review briefly in Section 2 some of the methods of recognizing and qualifying temporal chaos which may also be of value for STC. In Section 3 we briefly summarize work in the field of spatial chaos, whilst in Section 4 we address a classic example of spatio-temporal complexity—turbulent fluid flow. Finally, in Section 6 we speculate on possible quantifiers for STC.

A note on our policy on references is necessary. The vigour of research in this area has led to a huge number of published papers. Our references should be taken as examples, many containing extensive bibliographies; in no way is our list exhaustive but merely (and subjectively) representative.

2. CHAOS IN TEMPORAL SYSTEMS

The behaviour of a system which varies only with time is often summarized by one or more simple 'time series', in each of which the variation with time of a measurable variable is exhibited. Such a time series may have one of several broad, qualitative characteristics; it may be constant, or simply periodic (though even here 'the shape' of the periodic oscillation may take an infinite variety of forms). On the other hand it may be more complex, perhaps identifiable as multiply-periodic, chaotic or stochastic. A vast literature has grown over the last twenty years or so, concerned with methods of detecting and quantifying chaos, and we summarise here some results which will undoubtedly be of value in developing similar methodology for spatio-temporal chaos.

The most elementary analytic tool, the Fourier transform, enables us to distinguish between the various qualitative forms of behaviour; singly- or multiply-periodic behaviour corresponds to discrete spectra with peaks at the appropriate frequencies or sums/multiples of frequencies; chaos or stochasticity yield broad-band spectra. However, the distinction between deterministic (chaotic) and nondeterministic (stochastic) behaviour is difficult, or even impossible to make from spectra alone; the optimal use of information contained in the time series in order to make this distinction is the subject of much recent research [1–5].

The essential characteristic of chaos is that the time behaviour, though complicated, is deterministic, involving only a finite number of degrees-of-freedom, but is sensitively dependent on initial conditions, so that a small perturbation of the initial condition grows exponentially. At large time the evolution of a dissipative chaotic system carries it towards a (usually fractal) subspace of the full phase-space, 'a strange attractor'.

This divergence of neighbouring solution trajectories provides the basis for a quantifying tool, the set of Lyapunov exponents [6, 7], which measure the local rate of divergence. Thus, for example, if we are concerned with the solution to an equation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x} = [x_1, x_2, \ldots, x_n] \in U, \qquad \mathbf{f} = [f_1, \ldots, f_n]^T \tag{1}$$

where U is an open set in $\mathbb{R}^n$, and if TU_x is the tangent space to U at the point $x \in U$, then the tangent vector $\mathbf{y} \in TU_x$ satisfies the variational equation

$$\dot{\mathbf{y}} = \mathbf{A}\{\mathbf{x}(t)\}\mathbf{y} \tag{2}$$

where $\mathbf{A}(\mathbf{x})$ is the Jacobian matrix, given by $\mathbf{A}(\mathbf{x}) = \delta\mathbf{f}/\delta\mathbf{x}$. Now if we take an initial point $\mathbf{x}(0)$, and an initial perturbation $\mathbf{y}(0)$ in the tangent space $TU_{\mathbf{x}(0)}$, the maximum one-dimensional Lyapunov exponent is given by

$$\lambda_{\max} = \lim \frac{1}{t}\frac{y(y)}{y(0)} \tag{3}$$

where $\mathbf{x}(t)$ is a solution of (1) and $\mathbf{y}(t)$ is a solution of the variational equation (2).

The Lyapunov exponent is one of the most important measures by which to characterize a particular dynamics. Any system having at least one positive Lyapunov exponent is defined to be chaotic, i.e. if $\lambda_{max} > 0$. The magnitude of the Lyapunov exponent reflects the timescale on which the system dynamics becomes unpredictable, or on which transients decay.

When the information on the system is limited to one or more time series, with no knowledge of any generating equation or map, the calculation of Lyapunov exponents requires a vital preliminary step, the reconstruction of the phase-space from the time series [1]. This can be achieved by the method of delays [8, 9]; thus, if we have a single signal measured as a function of time, $\mathbf{x}(t)$, we sample the data at intervals τ_s, take a sample of M such values, and create a 'trajectory', $\mathbf{X}(t)$, of the system in M-dimensional space by constructing the vectors

$$\mathbf{X}_i(t) = \{x(t), x(t - \tau_d), \ldots x(t - (M - 1)\tau_d)\}$$

for a number n of discrete times, $t = i\tau_d$, $i = 1, \ldots, n$. Here we call M the embedding dimension, τ_d the delay time and τ_s sampling time; it is clear that τ_d must be an integer multiple of τ_s.

It is then known that [9], if the series is generated by the explicit system $\mathbf{X} = f(\mathbf{X})$, the points $\mathbf{X}_i(t)$ lie on a set diffeomorphic to the attractor, A, of the system. Using this approach we can deduce properties of A from the set of vectors $\mathbf{X}_i$.

Many authors have addressed the problem of optimising τ_d, τ_s and M in order to yield maximum information on the system. One of the more successful approaches, the singular value decomposition method [10, 11], is based on the Karhunen–Lorve decomposition theorem [12]. This procedure finds a set of orthogonal vectors spanning the embedding space. The number of orthogonal vectors forms an estimate of the dimension of the smallest space that contains a system's attractor. To find the delay time, τ_d, which makes $x(t)$ and $x(t + \tau_d)$ independent, this method uses the autocorrelation function, and suggests that the time at which the autocorrelation function passes through 0 is a suitable choice for τ_d. A rather similar method, which we might call the well-adapted basis method, has also been developed recently [13]. A third method, the mutual information method [14, 15], uses a different approach, dependent on the calculation of joint probability densities of information obtained at two times, T and $T + \tau_d$. Though the method is powerful in principle, it is difficult to apply in practice, since the estimation of joint probability densities is not straightforward; effectively the method needs very long time series to give reliable results.

Much recent effort has gone into predicting the future behaviour of nonlinear deterministic systems when a history of past behaviour is known, but when the underlying equations are either unknown or too complicated for direct solution.

The problem is essentially one of interpolation in phase-space on a basis of a swarm of neighbouring points (vectors in the embedding space arising from earlier samplings of the time series). The reader is referred to Smith [5] for an excellent exposition of this very recent approach.

Aside from the distinction between chaotic and stochastic behaviour, the character of the behaviour of a system depends crucially on its number of degrees-of-freedom, and especially on the character and distribution of its attractors. A number of quantitative tools are available, including Kolmogorov–Sinai entropy, and various information dimensions [16]. An excellent collection of such methods is found in Mayer-Kress [17], and many important advances have since been made [18, 21]. The possible extension of some of these approaches to the quantification of spatio–temporal chaos is discussed in Section 5.

3. SPATIAL PURELY STATIC CHAOS

The concept of spatial chaos is not yet as developed as that of temporal chaos. Indeed, the concept of temporal chaos is heavily dependent on the 'time-like' character of time-dependence; in other words, we have an evolutionary process. Based on some starting condition, we are concerned with behaviour for large time, t, or even, in some formal sense, as $t \to \infty$. Spatial phenomena are commonly confined within some domain Ω, with boundary $\delta\Omega$, on which appropriate boundary conditions are specified. However, although all spatial domains must be eventually finite, many configurations are so relatively extended that time-like behaviour in space may be possible over wide ranges. The most convincing example is the classical Euler elastica, in which the existence of chaotic loop sequences in a static configuration was conjectured by Holmes and Marsden [22] and subsequently explored more fully by Mielke, Holmes, Thompson, El Naschie, Kapitaniak, Moon and others [23–27]. Other examples from condensed matter physics (elastically coupled chains of atoms in a periodic potential) and biophysics (protein chains) have strong similarities, whilst branching or confluent static structures, as seen in biological morphogenesis or rivulet patterns, constitute a different class of candidates for spatial chaos. A recent review by El Naschie [27] contains extensive references to these and other problems.

In the case of the elastica, it may be shown [28] that the ODE in the space variable describing the static equilibria of an elastica with periodic axial imperfections

$$\phi_{ss} + \sin\phi = a\sin s \tag{4}$$

where ϕ is the gradient of displacement, and s is arc length, is identical to the equation for a periodically excited simple pendulum, i.e.

$$\ddot{\Xi} + \sin\Xi = a\sin t \tag{5}$$

Not surprisingly, we see a spatially chaotic distribution of stationary twists or loops in experiments on very long (laboratory) or infinitely long (numerical) periodically imperfect elastica subjected to appropriate conditions on ϕ and ϕ_s at $s = 0$ as shown in Fig. 1.

Holmes [29] has studied the effects of small spatial or temporal perturbations of the Sine–Gordon equation

$$\dot{\phi} + \phi_{ss} + \sin\phi = F(s, \mathrm{T}) \tag{6}$$

for the case $F(s, t) = F(s)$, and subject to the boundary conditions

$$\phi_s = 0 \text{ at } s = \pm\infty. \tag{7}$$

It appears that purely spatial chaos is not seen (all stationary non-periodic solutions are unstable) and all "stable" solutions must have spatio–temporal variation.

When $F(s, t) = -\varepsilon\phi_t$, and we have the following form of boundary conditions

$$\phi_s(0) = EH,\ \phi_s(1) = E(H + I(t)) \tag{8}$$

it is again found that all stationary or periodic solutions are unstable (except that infinite sets of stable orbits with arbitrarily long periods are created at the global bifurcation which generates chaotic solutions.

Overall, the subject of purely spatial chaos is in an initial state, but motivation from problems ranging from shell buckling to protein folding supports the importance of ongoing research in this field [30, 31, 97–99].

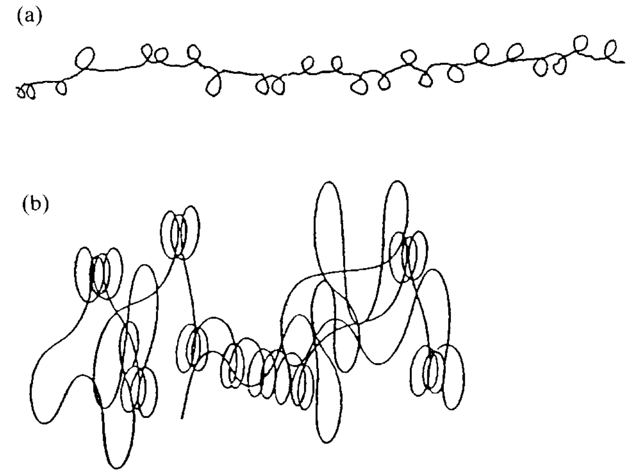


Fig. 1. The spatial plots, following ref. [26], for inperfect elastica; (a) $\phi_{ss} + \sin\phi = a\sin\omega s$; $\omega = 1$, $a = 0.01$, (b) $\phi_{ss} + \sin\phi + \delta\phi_s = a\sin\omega s\sin\phi$; $\delta = 0.15$, $\omega = 1.58$, $a = 0.94$.

4. AN EXAMPLE OF SPATIO-TEMPORAL COMPLEXITY: TURBULENT FLUID FLOW

Perhaps the most widely studied candidate for description as spatio-temporal chaos is turbulent fluid flow. Though no two fluid dynamicists agree on a formal definition of turbulence, there is wide agreement that it is characterized by a broad-band distribution of scales of motion in both time and space, and extreme sensitivity to initial conditions and parameter values. There is consequently a persistent faith amongst at least a section of the scientific community that some turbulent flows might be comprehensible in terms of solutions to a (relatively low-order) ODE representation of the full Navier–Stokes equations [32].

Whilst no formal reduction of the Navier–Stokes equations to an equivalent set of ODEs has been achieved (other than under the most demanding symmetry conditions), the observations in some real fluid systems of behaviour which is unmistakably similar to chaotic behaviour in a dynamical system sustains this faith. The most convincing example is the observation of period-doubling and other universal phenomena in Rayleigh–Bernard convection in a small box [33–35]; more recently evidence has been obtained of similar behaviour in Taylor–Couette flow with small aspect ratio. The suggestion of a chaotic regime preceding regimes of 'soft' and then 'hard' turbulence as the Reynolds number is increased. Quite recently, these two turbulence phases are reproduced with the use of coupled map lattice methods described in Section 5 corresponding to convection [96].

Intermittency is a phenomenon similarly shared between fluid mechanics and low-dimensional dynamical systems. Again the comparison is convincing in closed flows, less so in open flows in channels or boundary layers [37–39]. Most observations have been concerned with conditions near laminar–turbulent transition. Attempts to fully investigate turbulent flows, for example measuring dimension or Kolmogorov entropy at Reynolds numbers far above critical for transition, have had mixed results, which may be typified by analysis of atmospheric data over a variety of time-scales [40, 41]. In general, though the results apparently suggest that the dimension of the attractor is finite, it is nevertheless so large as to inhibit any transfer of concepts of the dynamics of low-order systems. This does not, of course, preclude the construction of very low-order conceptual models, linking through a set of nonlinear ODEs or difference equations the variation of integral properties of the system, e.g. models of ocean–atmosphere interactions predicting El Nino type phenomena [42]. Such models have often proved immensely valuable in alerting investigators to qualitative possibilities and in orientating more detailed programmes of research.

The most striking qualitative contrast of spatio–temporal behaviour of fluids occurs between flows which are closed and flows whcih are open. To a certain extent, the definition of closed and open is arbitrary, but essentially by a closed flow we mean a flow for which the constraints exerted by boundaries are so strong that all parts of the flow are instantly influenced by all other parts. Closed-flow structure is determined globally, and the system of fluid and boundaries is self-contained in the sense that no information can flow into or out of it. If the system is not closed it is open, and in this case flow structure can be local in character, instantaneously unrelated to structure in other regions but prey to unknown 'information' fluxes.

An alternative but similar classification into large-scale and small-scale flows has been proposed [43], using the concept of a correlation length L, based on a space correlation function

$$C(r - r') = \langle (u(r, t) - \langle u \rangle)(u(r', t) - \langle u \rangle) \rangle$$

which is presumed to vary like $\exp[-r/L_c]$ as $r \to \infty$.

A small system then has $L_c > L$, where L is a typical geometrical dimension; it may be regular or chaotic in time but coherent in space. The contrasting case in which $L_c < L$ displays behaviour incoherent in space; at moderately supercritical Reynolds number there may be 'spatial chaos, characterized by the chaotic evolution of coherent structures roughly of size L_c'. At a local level of observation a large-scale flow appears open, receiving external inputs of information.

Closed flows have been much studied because of their conceptual simplicity and because of the richness of behaviour they exhibit as the Reynolds number is increased and sequences of laminar patterns are eventually succeeded by a form of turbulence. Flow structure is dominated by global 'modes', dictated by the boundary constraints (e.g. Rayleigh–Bernard convection cells, Taylor vortices), and the dynamics of the modes, which may or may not correspond to separately identifiable flow features, can lead to a form of 'phase turbulence' in which time variations at a point are undoubtedly chaotic, but in which well-defined spatial structure still remains [44–48]. The energy of the flow may be concentrated in a small number of such modes, and it may be possible by a formal projection of the flow field onto a complete set of normal modes to obtain a low-order system whose dynamics models the flow quite well near to a point in parameter space of multiple bifurcation [49, 50].

Open-flow systems, by our definition, are not dominated by global modes, and flow structure is at least partly determined by local dynamics. Nevertheless, the occurrence of recognizable coherent structures in such flows is widely reported [see 51].

Pipe or channel flows which are laterally constrained but longitudinally open, and therefore open to influence from the input of information from 'upstream', received early attention, especially in respect of the occurrence of intermittency and its relationship to intermittency in simple dynamical systems. Later studies, especially by Sreenivasan and several co-workers, developed these ideas in the context of multi fractal structures, (see [52]).

Another open flow has inspired imaginative (and successful) attempts to obtain, by direct use of observational data, a sequence of empirical eigenfunctions whose evolution is determined by a set of coupled nonlinear ODEs [53, 54]. This work was, however, concerned with the wall region of a turbulent boundary layer, in which structures are strongly influenced by the geometrical constraint of the wall. More truly open flows, such as wakes or other free shear layers, may prove less amenable to such analysis.

The intrinsic intractability of the Navier–Stokes equation has motivated a search for simpler extended systems in which spatio–temporal phenomena may be studied. We may replace the continuum by discrete distribution of nodes, each of which is specified by a 'state' or 'phase' which evolves according to a discrete time-map. These nodes may be chosen to have interactions of diffusive or advective nature, local or global, so as to form a coupled map lattice (CML) and we discuss this model in detail in the next section.

5. SPATIO–TEMPORAL CHAOS IN COUPLED MAP LATTICES

The concept of a coupled map lattice (CML) has been useful in studying spatio–temporal chaos. A CML is a dynamical system with discrete time ('map'), discrete space ('lattice'), and a continuous state. It usually consists of dynamical elements on a lattice each interacting ('coupled') among suitably chosen sets of other elements [55–73, 78–80, 83–87].

The modelling of a dynamical phenomenon with spatial structure through a CML is carried out as follows. We first decompose its dynamics into simple procedures, and then replace each procedure by a parallel dynamics on a lattice. The coupled map lattice dynamics is then investigated by carrying out each procedure successively. Schematically it can be written as

$$x_n(i) \to x_n(i) = F_1[\ldots, x_n(i-1), x_n(i), x_n(i+1), \ldots],$$

$$x_n(i) \to x''(i) = F_2[\ldots, x'_n(i-1), x'_n(i), x'_n(i+1), \ldots]$$

$$\ldots$$

$$x''^{\cdots\prime} \to x_{n+1}(i) = F_k[\ldots, x''^{\cdots\prime}_n(i-1), x''^{\cdots\prime}_n(i), x''^{\cdots\prime}_n(i+1), \ldots] \qquad (9)$$

by using k successive procedures F_j, $j = 1, 2 \ldots, k$.

Here i is a spatial lattice point and n is a discrete time-step. Note that the lattice spacing and temporal unit are not microscopic but finite sized; $x_n(i)$ is a coarse-grained quantity at this 'semi-macroscopic' level.

As an example, we might attempt to model some phenomenon in a fluid, specified by a nonlinear process and diffusion. In the CML approach we decompose the dynamics into local evolution and spatial diffusion processes. As a simple choice we adopt a logistic map for the local behaviour

$$x'_n(i) = f\{x'_n(i)\},\ f(y) = 1 - ay^2,$$

and a discrete Laplacian operator for the diffusion

$$x_{n+1}(i) = (1-\varepsilon)x'_n(i) + (\varepsilon/2)\{x'_n(i+1) + x'_n(i-1)\}.$$

Combining the above two processes our dynamics is given by

$$x_{n+1}(i) = (1-\varepsilon)f\{x_n(i) + (\varepsilon/2)[f\{x_n(i+1)\} + f\{x_n(i-1)].\qquad (10)$$

the above CML has been investigated extensively as a standard model for spatio–temporal chaos. We stress that the local evolution and spatial diffusion processes are carried out separately; this is the key simplifying feature of CMLs. In studies of CMLs we search for novel qualitative universality classes of behaviour, without worrying about the details of phenomenology. We expect that such universality classes will exist, and will eventually constitute the language for describing and classifying STC more generally. It is convenient to develop our ideas on the basis of equation (10).

5.1. *Phenomenology of spatio–temporal chaos as exhibited in a CML*

In the model (10), successive transitions lead from frozen random pattern to pattern selection to spatio–temporal intermittency, and finally to fully developed spatio–temporal chaos [58]. This class of successive changes is found in a large class of spatially extended dynamical systems with spatial symmetry, giving support for our search for qualitative universality.

5.1.1. *Frozen random pattern.* The CML (10) exhibits period-doubling of kinks with increase of the 'nonlinearity'. As a result of the doublings, domains of various sizes are formed. After some number of doublings the system has a chaotic appearance. Because of the sensitive dependence on initial conditions, a homogeneous state is unstable and a domain structure is spontaneously created even if we start from an almost homogeneous initial condition (see Fig. 2(a)). The frozen random pattern leads to spatial bifurcation. Even if the model is homogeneous in space, attractors can have strong spatial dependence. In a large domain, the motion is quite chaotic, while it is almost period-eight at smaller domains, period-four for much smaller domains, and period-two for the smallest ones. Distribution of domain sizes can differ by initial conditions. We can choose initial conditions so that attractors have an arbitrarily large domain. In general we expect that the number of attractors increases exponentially with the system size.

5.1.2. *Pattern selection with suppression of chaos.* As the nonlinearities increase further, larger domains start to be unstable and split into smaller domains. Initial conditions are no longer remembered (Fig. 3), and, through the transient process, domains of a few special sizes are selected. After the selection the pattern of domains is frozen and does not move in space. Selected sizes of domains are such that the dynamics of the domains is less chaotic within the frozen random pattern. This process may be understood in the sense that the diffusion tries to homogenize a system, while the chaotic motion makes the system inhomogeneous because of the sensitive dependence on initial conditions. These two

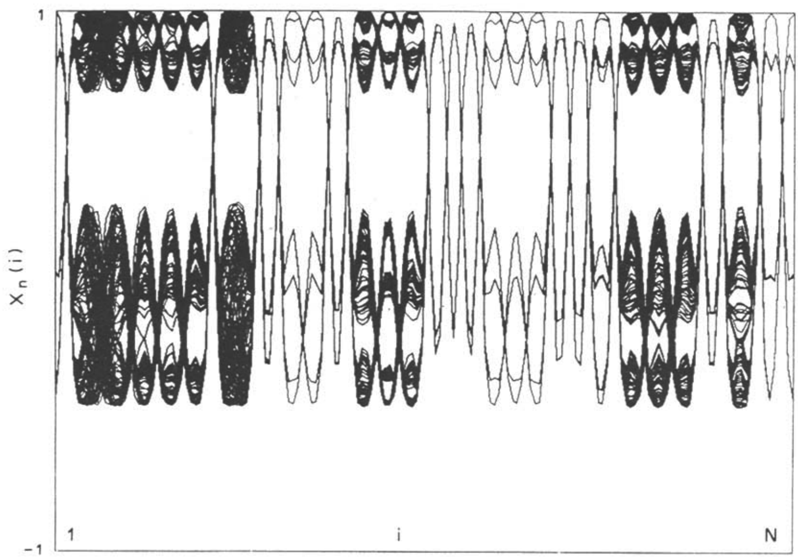
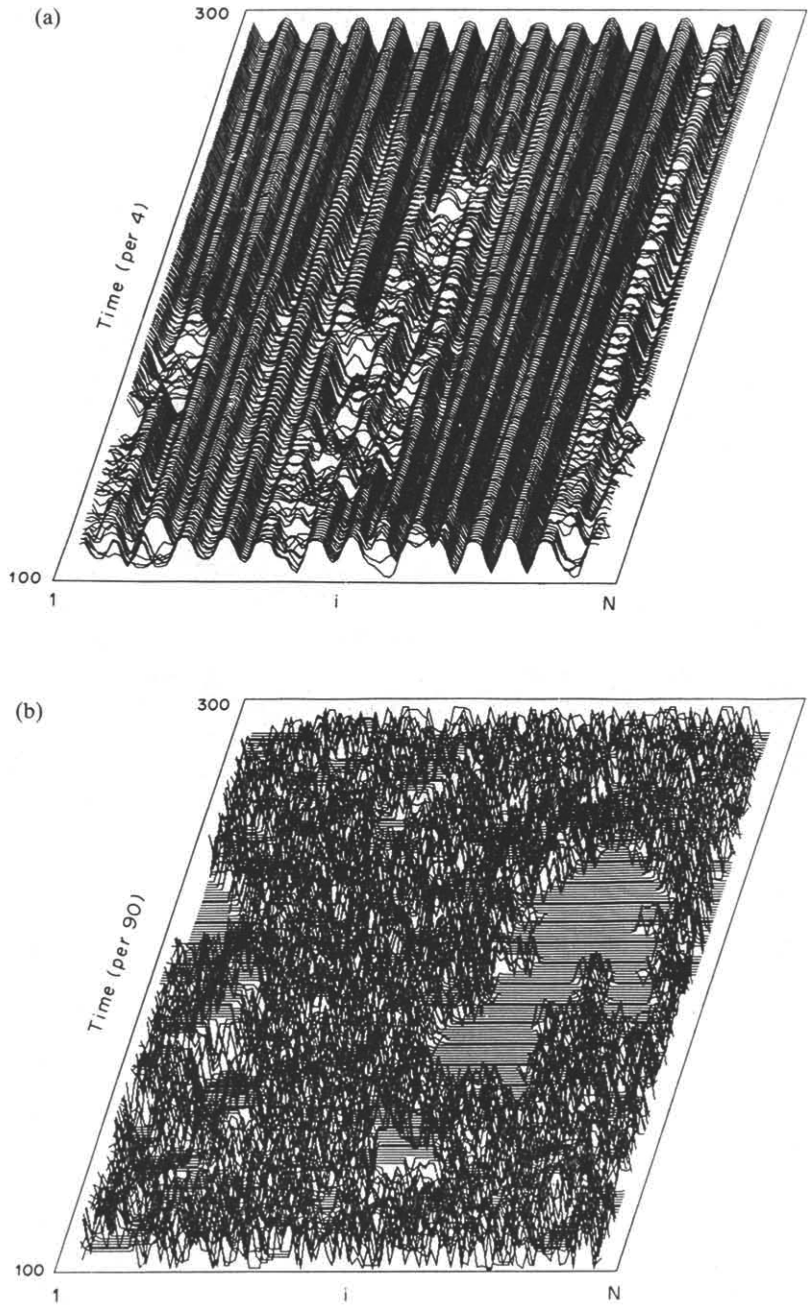


Fig. 2. Space–amplitude plot for the coupled logistic lattice (10). Amplitudes $x_n(i)$ are overlaid for 1000 time steps after discarding 100 000 transients, starting with random initial conditions; $\varepsilon = 0.4$, $a = 1.46$, $N = 160$.

tendencies conflict with each other. In a large domain the chaos is so strong that it splits into smaller domains (one may regard this as splitting by the 'chaos pressure'). Once a domain structure is formed with the suppression of chaos, the conflict is resolved, and the domain structure is stabilized. This picture leads to the conjecture that a pattern with smaller Lyapunov exponents is selected. Numerical results seem to support this conjecture. The simplest example of pattern selection in system (10) is the selection of a zigzag pattern



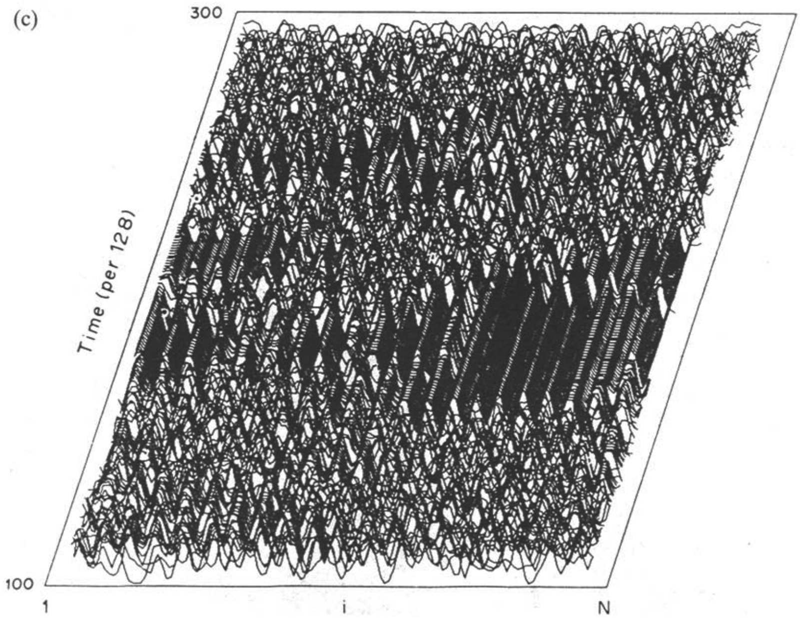


Fig. 3. Plot of $x_n(i)$ as a function of space. 200 sequential patterns $x_n(i)$ are depicted with time (per some time steps), starting from random initial condition. Unless otherwise mentioned, the system size N is chosen to be 100. (a) $a = 1.66$, $\varepsilon = 0.4$, plotted per 4 time-steps, after discarding 400 initial transient steps, (b) $a = 1.76$, $\varepsilon = 0.005$, plotted per 90 time-steps, after discarding 90000 initial transient steps, (c) $a = 1.735$, $\varepsilon = 0.35$, plotted per 128 time-steps, after discarding 90000 initial transient steps.

(domain size = 1, i.e. $k = 1/2$). There can be two regions of the zigzag pattern with different phase of oscillation. As time goes to infinity, a single domain of zigzag pattern covers the whole space. In the transient time regime, we have seen defects as a domain boundary between two zigzag patterns with different phases of oscillation. The defect is localized but moves in space; the motion of defects is chaotic in time, as is verified by positive Lyapunov exponents. Defects pair-annihilate, and the domain size of a connective zigzag region increases with time. The motion of defects is well described by Brownian motion. Indeed there exists a well-defined diffusion constant from numerical results. We have also estimated Kolmogorov–Sinai (KS) entropy for the defect from the sum of positive Lyapunov exponents. Roughly speaking, the diffusion constant of a defect increases proportionally with its KS entropy as the nonlinearity is increased. KS entropy gives the rate of memory in the phase-space. If our diffusion is triggered by the chaos of a defect, the present Brownian motion can be represented roughly by a rate of 'coin tossing' per some 'memory' time which is inversely proportional to the KS entropy. Our observation of the proportionality between KS entropy and diffusion constant supports the above idea.

5.1.3. *Spatio–temporal intermittency.* Transition from an ordered pattern to fully developed spatio–temporal chaos occurs via spatio–temporal intermittency (STI). In STI there coexist laminar motion and turbulent bursts in space–time. Each space–time pixel can be classified into laminar (L) and bursts (B). Since the recognition of STI in 1984 [55], studies have been growing both experimentally and theoretically. So far, two types of STI have been recognized. In the first type [55, 64, 73], there is no spontaneous creation of bursts; if a site and its neighbours are laminar, it is still laminar in the next step. Before the onset of

STI a spatially homogeneous, temporally periodic state is stable. Possible relationships of type-1 STI with directed percolation have been intensively investigated. Results are similar qualitatively, but there seems to be a quantitative difference [64]. The first example for this type is given in the coupled logistic lattice (10) at a parameter region for period-three window and very weak coupling (see Fig. 3(b)). In the second type of STI there seems to exist [58, 63] spontaneous creation of turbulent bursts as long as some coarse-grained reduction of states is used. There is some probability of creation of bursts even if all the states of a site and its neighbours are laminar. It might be possible to introduce other states between laminar and bursts so that no spontaneous creation of bursts from the laminar states is possible, but it is not yet clear whether such partition is possible for only a finite number of states. This STI is observed in transitions with a state spatial pattern (Fig. 3(c)). Even before the onset of STI there is a spatial structure as in the second case above. So far this type of STI is observed as a transition from local to global chaos. In type 2 STI, the temporal change corresponding to the selective pattern has a very long memory, leading to selective-flicker noise. The dynamical form factor $P(k, \omega)$ (power of Fourier transform of the space–time pattern $x_n(i)$) exhibits $\omega^{-\beta}$ noise ($\beta \approx 1.9$) only for the wavenumber $k \approx k_{\mathrm{p}}$, the wavenumber of selected pattern [58]. It is worth remarking that phenomena identifiable as type-2 STI have recently been observed in various experiments with fluids. In all the examples the transition is associated with a spatial structure (a selected wavenumber), and includes spontaneous creation of turbulent states from a laminar region. Examples include instances of Bernard convection [88, 90], and the Faraday instability of a wave [91]. In two-dimensional electric convection of a liquid crystal, STI has been found at the collapse of selected chequerboard patterns [89]. The transition is again chaos/chaos transition admitting spontaneous creation of turbulent states. Flicker-like noise, with $P(k_{\mathrm{p}}, \omega) \approx \omega^{-1.9}$, for the wave number k_{p} corresponding to the chequerboard pattern, is again found. Another related phenomenon associated with the onset of global turbulence is soliton turbulence, first found in a coupled circle lattice [59]. In lattices of circle maps,

$$f(x) = x + \Omega + [K/2\pi]\sin(2\pi x),$$

there is a kink structure which propagates with a constant velocity. At the onset of global turbulence interactions of kinks can create turbulent bursts, or a nucleus emitting kinks. The motion is turbulent but it consists of propagation of kinks and their interactions. See also [82] for soliton turbulence in cellular automata.

5.1.4. *Quasi-stationary supertransients and fully developed spatio–temporal chaos.* In low-dimensional dynamical systems chaos is structurally unstable, and small windows of nonchaotic behaviour are interspersed in any parameter regime. In fully developed spatio–temporal chaos we do not, in general, observe such window structures, despite the fact that the homogeneous state with a stable cycle corresponding to a window is linearly stable also in a coupled system. Of course, if we start from the vicinity of a homogeneous state, our system is attracted into that state within a finite number of time-steps. The volume of suitable initial conditions, however, decreases very rapidly with the system size (roughly exponentially). As an example, we have examined whether fully developed spatio–temporal chaos is really the 'ultimate attractor' for our logistic lattice at a parameter corresponding to the period-3 window. For small couplings and lattice size we have always observed an escape from a chaotic state to the homogeneous periodic state. The transient time for the escape, however, diverges exponentially with the system size, so that if the system size is, for example, larger than ten lattice sites, it is practically impossible to wait and see if the system really hits the homogeneous attractor. Such very long transients, which we call 'supertransients' are often encountered in spatio–temporal chaos. In the supertransient regime behaviour is quasi-stationary without any symptoms of decay of quantities characterizing spatio–temporal chaos, such as Lyapunov exponent, KS entropy, and dimension. It is almost impossible to forecast when a transient will terminate; furthermore, the quasi-stationarity makes it almost impossible to distinguish transients from attractors. We might argue that the 'stability' of fully developed spatio–temporal chaos is sustained by a product of supertransients of this type. It is worth remarking here that Rossler has introduced the term 'hyperchaos' as chaos in which the number of positive Lyapunov exponents is more than one. Extending his idea we might speculate that fully developed spatio–temporal chaos may be described as (hyper)chaos in space; a direct product of many chaotic systems. The idea of this construction originates in the synthesis of Landau's picture of turbulence and hyperchaos [70]. Landau has tried to understand turbulence as a direct product of periodic states (leading to a quasi-periodic state with many incommensurate frequencies). This direct product state is, however, unstable because of frequency lockings and nearby strained attractors [56, 75]. On the other hand, a turbulence model as a direct product of chaotic states (hyperchaos) is structurally stable by the above mechanism.

5.1.5. *Spatial bifurcation in open flow models.* Sections 5.1.1–5.1.4 have described the phenomenology in a system with spatial symmetry; this has some similarities to a closed fluid flow. In the open system (like a pipe flow) the coupling is asymmetric; there is a strong influence from the upstream direction, as represented by the $\delta/\delta x$ term in partial differential equations. In our CML model, open flow is easily simulated by spatially asymmetric coupling. Here we take the extreme limit, a one-way coupled model

$$x_{n+1}(i) = (1 - \varepsilon)f\{x_n(i)\} + \varepsilon f\{x_n(i - 1)\}. \tag{11}$$

This model exhibits spatial period doubling [60, 61] and selective amplification of noise. Its dynamical state changes from fixed point to period-2, period-4, . . . successively as the lattice point goes downstream (Fig. 4(a)). After some doublings, the system goes to a turbulent state. Such spatial bifurcation is also found in experiments in pipe flow. As the nonlinearity is increased, successive changes among (a) flow with randomly chosen patterns, (b) flow with selective patterns, (c) transmission of defects, (d) spatio–temporal intermittency, (d) fully developed spatio–temporal chaos are observed. This one-way coupled model provides simple abstraction of the open fluid flow discussed in Section 4.

Extension of the diffusively coupled map lattice to higher dimensions is straightforward. In a two-dimensional system we have again observed frozen random pattern, pattern selection, spatio–temporal intermittency, and fully developed spatio–temporal chaos [84]. In a larger coupling, or in a higher dimension frozen domain, structures are unstable and the formation of spatial structure is more difficult. For a lattice with a dimension greater or equal to 2, it is expected that there is an upper bound on the coupling strength, beyond which any frozen domain pattern loses its stability. Indeed, even in a one-dimensional lattice, we often encounter a floating domain if the coupling is very large.

In an infinite dimensional lattice, i.e. in a mean-field coupled model, we have a novel class of dynamical transitions which has been called clustering. It is an open question if there is a critical dimension for the mean-field behaviour in our CML.

6. QUANTIFIERS FOR SPATIO–TEMPORAL CHAOS

Traditionally, we often use Fourier transforms in space and time to characterize spatio–temporal patterns. Power spectra of Fourier transforms in space/time (dynamical form factor) are still useful to study spatio–temporal chaos. In particular, the appearance of

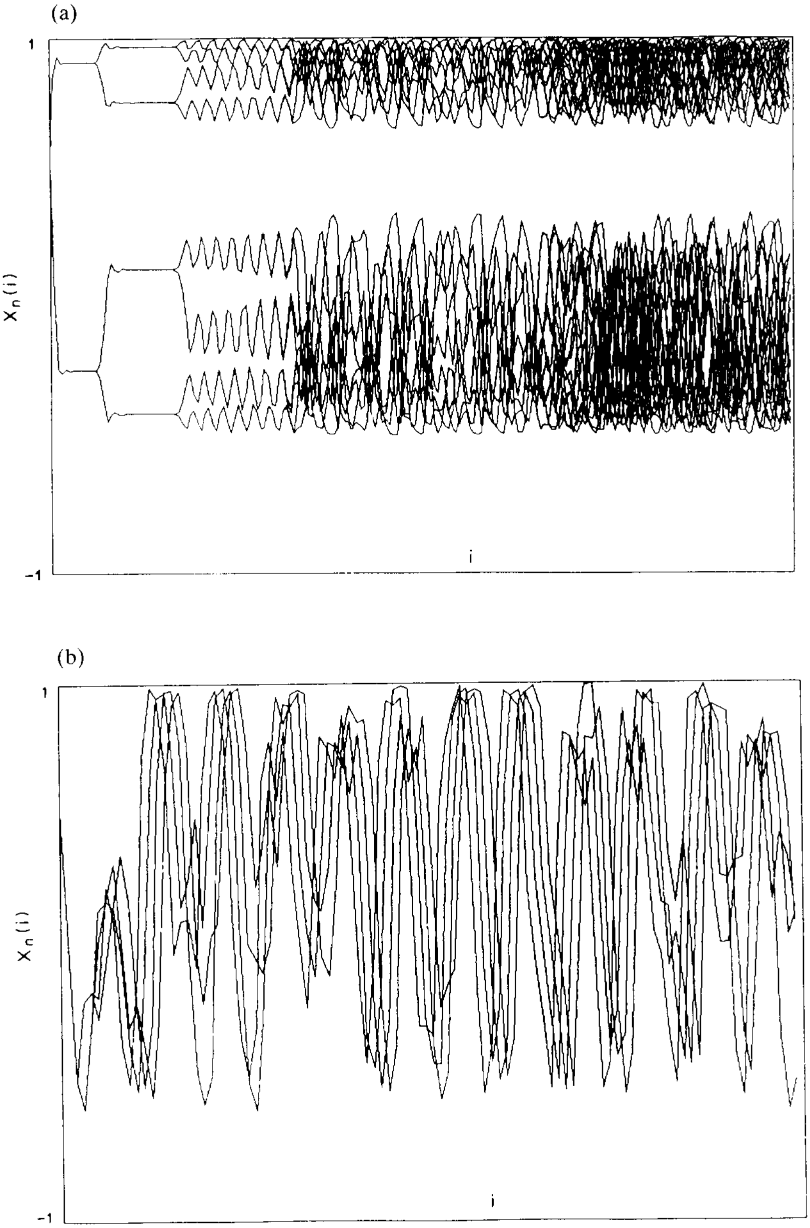


Fig. 4. Space–amplitude plot for the one-way coupled logistic lattice (11); $\varepsilon = 0.3$. Amplitudes $x_n(i)$ are overlayed, after discarding 10 000 transients, starting with a random initial condition; (a) $a = 1.48$, $N = 200$, overlaid over 1000 steps, (b) $a = 1.6$, $N = 100$, overlaid over 4×4 steps per 4 steps.

long-range correlations in STI is characterized by power-law behaviour in power spectra. Also phase changes of patterns may be studied by the use of order parameters corresponding to pattern dynamics [58].

Besides these traditional quantifiers, it is interesting to extend quantifiers of dynamical systems to spatial systems. We may then see how spatio-temporal chaos may be understood from the terminology of dynamical systems, and also establish relationships with traditional quantifiers. To complete that task we need a thermodynamic theory of spatio–temporal chaos which is still missing; here we attempt to take some first hesitant steps.

(a) Lyapunov analysis gives information on the tangent space of an orbit [55, 57, 58]. The Lyapunov spectrum is a measure of how a small deviation expands with a chaotic orbit. With the pattern changes described in Section 5, the spectrum shape changes from step-like (a frozen random pattern), to concave (at intermittency), and then convex (for fully developed spatio–temporal chaos) as the nonlinearity is increased. This convex shape is in strong contrast with the spectra for the Kuramoto–Sivashinski equation [93] and for the Gledzer model for turbulence [94]. From the spectra, the density of KS entropy and Lyapunov dimension are obtained [57]. Since the spectrum has a size-invariant form when suitably scaled [57], the density of entropy and dimension are well defined. This density gives a size-independent measure for the strength of chaos. The Lyapunov vector (eigenvector for the spectrum) gives eigenmodes with different directions of instability. Lyapunov vectors corresponding to chaotic modes are localized in real space through a mechanism similar to Anderson localization [57]. To distinguish laminar and turbulent regions in space–time, subspace–time Lyapunov spectra [72] are introduced as the extension of Lyapunov analysis to local space–time patches. The subspace–time Lyapunov exponents measure the degree of instability in a space–time patch. By sampling these exponents we can construct a distribution function of subspace–time. In STI, the distribution is clearly separated between positive and negative parts, whilst in fully developed spatio–temporal chaos it approaches Gaussian form.

(b) To measure the amplification of a moving disturbance in an open flow (convective instability) it is useful to introduce co-moving Lyapunov spectra [57, 61, 67]. They are the spectra in a Galilean frame moving with a finite velocity. The spectra are especially important in convective chaos [61] and also useful in analysis of the flow of information. In an open-flow system we have to distinguish absolute instability from convective instability. If a small perturbation against a reference state grows in a stationary frame we speak of 'absolute instability', while if the perturbation grows only in some frames moving with finite velocities we speak of 'convective instability'. In spatio–temporal chaos with open flow, a system often shows only convective instability. In this case the conventional Lyapunov exponents are negative, even if the spatio–temporal change in the variable is clearly chaotic. Only within a certain band of velocity $v_L < V < v_u$, does the co-moving Lyapunov exponent take a positive value [61]. This co-moving Lyapunov exponent clearly characterizes chaos in open flow. Co-moving Lyapunov exponents are also useful in estimating the propagation speed of a small disturbance [57] from a lattice point to other lattice points. Only within a band of velocity giving a positive co-moving Lyapunov exponent will a disturbance be propagated with amplification.

(c) Chaos is a source of information, as first clarified by Shaw [95]. Spatio–temporal chaos has the ability of information creation and selective transmission through space [57]. Co-moving mutual information flow is introduced to measure how information flows in space–time even in turbulent media. In soliton turbulence the propagation of information by solitons can be confirmed through this quantifier. Even in fully developed spatio–temporal chaos there remains some finite information flow [87].

(d) In low-dimensional chaos a dimension algorithm has been used as a standard diagnostic technique to distinguish chaos from random data. In spatio–temporal chaos, the dimension itself is an extensive quantifier and its density is more important. Using multi-point measurement, it is possible to estimate the dimension density from experimental data [67]. Singular value decomposition with Kahuren–Lowe technique may be useful for practical applications [10–12].

Theoretical formulation for these quantifiers has just been started, and most problems are left for the future. Bunimovich and Sinai [66] have constructed a statistical mechanical formulation for CMLs. Their theory is so far limited to fully developed spatio–temporal chaos in CMLs with complete hyperbolicity. It is a future problem in mathematical physics to formulate our phase-transition in pattern dynamics within the terms of statistical mechanics.

Finally, a self-consistent argument for the distribution of patterns has recently been formulated with the use of the Perron–Frobenius operator [68, 72]. It is possible to estimate the onset parameter of STI to this self consistent approximation.

Data from CMLs will provide an ideal test bed for these possible quantifiers of STC before they are used for the much more difficult purpose of characterizing universal behaviour classes (if they exist) in more complex physical situations like turbulent flows. The (distant) objective is that of predicting classes of universal behaviour in known systems, and, conversely, that of recognizing crucial underlying physics from spatio–temporal data, an end not yet achieved satisfactorily even for purely temporal systems.

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